3.2 Skew field or Division ring :-

A ring with at least two elements is called a skew field if (i) It has unity element
(ii) each non-zero element possesses multiplicative inverse.

Thus, A Commutative Skew field is a field

• Example of a skew field which is not a field.

Ex-(1) Prove that the set of all 2x2 matrices of the form $\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ where a, b, c and d are real numbers is not a field.

Soln:- Let M be the set of all 2x2 matrices of the form $\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ where a, b, c, d are real numbers.

First, we shall prove that M is a ring w.r.to addition and multiplication.

Let $A, B \in M$ where $A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$, $B = \begin{bmatrix} p+iq & r+is \\ -r+is & p-iq \end{bmatrix}$

Then, we have, $A+B = \begin{bmatrix} (a+p)+i(b+q) & (c+r)+i(d+s) \\ -(c+r)+i(d+s) & (a+p)-i(b+q) \end{bmatrix}$

which is obviously matrix of the given form $A+B \in M$

Also, we can show that

$A \in M,\ B \in M \Rightarrow AB \in M$

$\therefore M$ is closed w.r.to addition and multiplication.

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Further matrix addition is Commutative and Associative.

The zero matrix $\begin{bmatrix} 0+i0 & 0+i0 \\ -0+i0 & 0-i0 \end{bmatrix}$ is the additive identity and so, it is the zero element of M.

If $A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix} \in M$ then oviously $-A = \begin{bmatrix} -a-ib & -c-id \\ c-id & -a+ib \end{bmatrix} \in M$

Then each element of M possesses additive inverse.

Further, it can be shown that matrix multiplication is associative and it is distributive.

Hence, M is a ring w.r.to addition and multiplication of matrices.

- Existence of multiplicative identity:-

The matrix $\begin{bmatrix} 1+i0 & 0+i0 \\ -0+i0 & 1-i0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in M$

It is a unit matrix and so, it is the multiplicative identity.

Thus, M is a ring with unity.

- Existence of multiplicative inverse of each non-zero element of M.

Let $A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ be non-zero element of M

i.e. a, b, c, d are not zero.

we have,

$$|A| = \begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$$

$$= |(a+ib)(a-ib) - (c+id)(-c+id)|$$

$$= a^2+b^2+c^2+d^2 \neq 0$$

$\therefore$ The matrix A is non-singular and hence its inverse exists.

We must show that $A^{-1} \in M$

we have, $A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{|A|}\begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix} \in M$

Therefore, each non-zero element of M possesses multiplicative inverse.

$\therefore$ M is a skew field or division ring.

Now, in order to show that M is a field. we need to verify one more postulate namely Commutative of multiplication.

we shall show that M is not commutative. Hence, M is not a field.

For, this let us consider $A = \begin{bmatrix} 3+4i & 5+6i \\ -5+6i & 3-4i \end{bmatrix}$ and $B = \begin{bmatrix} 1+i0 & 1+i0 \\ -1+i0 & 1-i0 \end{bmatrix}$

Obviously, $A, B \in M$

Now, $AB = \begin{bmatrix} -2-2i & 8+10i \\ -8+10i & -2+2i \end{bmatrix}$ and $BA = \begin{bmatrix} -2+10i & 8+2i \\ -8+2i & -2-10i \end{bmatrix}$

we find that $AB \neq BA$

Therefore, multiplication in M is not commutative

Hence, M is not a field Proved